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Orientifolds, RR torsion, and K-theory

Oren Bergman

California Institute of Technology, Pasadena CA 91125, USA and
CIT/USC Center for Theoretical Physics
Univ. of Southern California, Los Angeles CA, USA
E-mail: bergman@theory.caltech.edu

Eric Gimon

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton NJ 08540, USA
E-mail: gimon@sns.ias.edu

Shigeki Sugimoto

CIT/USC Center for Theoretical Physics, Univ. of Southern California
Los Angeles CA, USA
E-mail: sugimoto@citusc.usc.edu

Abstract: We analyze the role of RR fluxes in orientifold backgrounds from the point of view of K-theory, and demonstrate some physical implications of describing these fluxes in K-theory rather than cohomology. In particular, we show that certain fractional shifts in RR charge quantization due to discrete RR fluxes are naturally explained in K-theory. We also show that some orientifold backgrounds, which are considered distinct in the cohomology classification, become equivalent in the K-theory description, while others become unphysical.

Keywords: D-branes, Differential and Algebraic Geometry.

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Contents

1. Introduction

Orientifold planes are intriguing objects in many ways. They appear, at least perturbatively, to be void of dynamics, and in this respect they are similar to orbifolds. On the other hand orientifold planes are also similar to D-branes, in that they appear with the same dimensionalities, carry RR charge, and have a non-vanishing tension.

(H, G_{6-p})	Op variant	Dp gauge group	charge
$(0,0)$	Op^-	SO$(2n)$	-2^{p-5}
$(1/2,0)$	Op^+	U Sp$(2n)$	$+2^{p-5}$
$(0,1/2)$	$\widetilde{\mathrm{O}p}^-$	SO$(2n+1)$	$-2^{p-5}+1/2$
$(1/2,1/2)$	$\widetilde{\mathrm{O}p}^+$	U Sp$(2n)$	$+2^{p-5}$

Table 1: Orientifold plane variants and their charges.

However the latter two differ from their D-brane values in two important ways. In some cases the tension of an orientifold plane is negative, and in some cases its charge (and tension) is fractional.

There are actually a variety of orientifold planes for each dimensionality p, associated with the possibility of turning on discrete fluxes of certain anti-symmetric tensor fields [1, 2]. These fluxes come in two varieties corresponding to the NSNS three-form H, and the RR $(6-p)$-form G_{6-p}, and have traditionally been classified by the (integral) cohomologies $H^3(\mathbb{RP}^{8-p})$ and $H^{6-p}(\mathbb{RP}^{8-p})$, respectively. The space $\mathbb{RP}^{8-p}$ is the "sphere" that surrounds the Op-plane in the reduced space of the orientifold projection. Except for the case $p=6$, both of these cohomologies are $\mathbb{Z}_2$, so the fluxes are torsion classes. This means that there are four variants of the orientifold plane for each p (for $p \leq 5$), depending on the values of the discrete fluxes (table 1). For $p < 2$ there are actually additional variants, which are associated with a discrete G_{2-p}-flux [2].

One immediately observes that orientifold planes with $p \leq 4$ are fractionally charged (relative to Dp-branes). This appears to be at odds with Dirac's quantization condition, suitably generalized to higher rank anti-symmetric tensor fields, or equivalently with the idea that RR charges and fields take values in integral cohomology. In particular, since the Op-plane is charged under C_{p+1}, the relevant cohomology group is given by $H^{8-p}(\mathbb{RP}^{8-p}) = \mathbb{Z}$. The generator of this group has a unit Dp-brane charge, so the Op-plane charge $\pm 2^{p-5}$ is not an allowed value for $p \leq 4$. In addition, the presence of RR torsion in $H^{6-p}(\mathbb{RP}^{8-p})$ shifts the charge by 1/2 for the Op^- plane, but not for the Op^+ plane. So in the first case it seems to lead to a further violation of Dirac's condition.

In some cases the violation can be attributed to an anomaly in the path integral for the dual magnetic object, i.e. a D$(6-p)$-brane, and can be expressed in terms of spacetime fields [3], but this is not generally the case. For example, the $\pm 1/2$ charge of the O$4^\pm$ plane is consistent with an anomalous phase in the D2-brane path integral coming from the fermion determinant [3]. However one needs another contribution to the phase in order to explain the 0 charge of the $\widetilde{\mathrm{O}4}^-$ variant, and yet another to explain the $+1/2$ charge of $\widetilde{\mathrm{O}4}^+$. One could imagine that the latter is due to the term $\int B \wedge C_1$ in the D2-brane action, but there is no obvious candidate for the former contribution. For other orientifold planes one does not even have an analog of the anomaly contribution of [3].

All this indicates that integral cohomology is not the appropriate mathematical framework for RR fields. We are familiar with the idea that D-branes, or more properly RR charges, take values in K-theory rather than in integral cohomology [4, 5, 6]. This follows from the fact that a Dp-brane is characterized not only by its charge under the RR field C_{p+1}, but also by a gauge bundle, which in turn determines its charges under lower rank RR fields. It has recently been suggested that the RR fields take values in K-theory as well [7, 8]. This is intuitively obvious for RR fields created by D-brane sources, since the field at infinity should belong to the same class as the source which creates it. But the idea is that all RR fields, including source-free RR fluxes, are classified in K-theory.

In this paper we investigate some of the consequences of describing RR fluxes in K-theory rather than integral cohomology. In particular, this provides a partial explanation of the observed fractional charges; K-theory correlates different RR fields in such a way, that the discrete flux of G_{6-p} alters the quantization of G_{8-p}-flux. For $p = 6$ the former (G_0) is integral rather than discrete, but there is an analogous correlation between the parity of G_0 and the half-integral part of G_2. Thus we are able to compute the shift in the quantization condition of the RR charge due to the discrete RR flux. We are not able to establish the quantization condition for the absolute charges however. In comparing the K-theory classification with the cohomological one we also discover that some discrete fluxes are actually trivial, while others are obstructed. This has interesting physical implications concerning some of the lower orientifold planes.

The rest of the paper is organized as follows. In section 2 we review what is known about orientifold planes, and their classification in terms of discrete fluxes in cohomology. In section 3 we describe the necessary K-theory machinery required for describing RR fields in orientifold backgrounds, and compute the relevant groups. We also explain how to compare the K-theory results with cohomology using a real version of the Atiyah-Hirzebruch spectral sequence (AHSS), which is further elaborated in the appendix. In section 4 we discuss three implications of the difference between K-theory and cohomology in orientifolds. The first concerns the question of proper RR charge quantization, the second is the identification in K-theory of certain orientifold variants which are distinct in cohomology, and the third is the dismissal of certain orientifold variants as unphysical. Section 5 is devoted to a separate discussion of the orientifold 6-plane, which in many ways was the original motivation for this work.

2. Review of orientifolds

2.1 Basics

Orientifolds are defined by gauging a discrete symmetry in string theory which includes a reversal of the string world-sheet orientation (denoted Ω). The loci of points which are fixed under this symmetry are called orientifold planes. In particular, an orientifold p-plane (Op) in type-II string theory is given by the $\mathbb{Z}_2$ projection

$$\mathrm{O}p: \qquad \mathbb{R}^{1,p}\times\mathbb{R}^{9-p}/\mathcal{I}_{9-p}\,\Omega\cdot\begin{cases}1 & p=0,1 \quad \mod 4\\ (-1)^{F_L} & p=2,3 \quad \mod 4\,,\end{cases} \tag{2.1}$$

where p is even in IIA and odd in IIB. The presence of $(-1)^{F_L}$ for $p=2,3 \bmod 4$ can be understood from the requirement that the generator square to 1 on fermions. For example, $(\mathcal{I}_2\Omega)^2=-1$ on fermions, but $(\mathcal{I}_2\Omega(-1)^{F_L})^2=(\mathcal{I}_2\Omega)^2(-1)^{F_L+F_R}=+1$. The action of the orientifold on the antisymmetric tensor fields is given by

$$\begin{aligned} B &\longrightarrow -B \\ C_{p'} &\longrightarrow +C_{p'} \qquad p'=p+1 \bmod 4 \\ C_{p'} &\longrightarrow -C_{p'} \qquad p'=p+3 \bmod 4\,. \end{aligned} \tag{2.2}$$

In particular, the projection leaves the Dp-brane invariant, and in fact preserves the same 16 supersymmetries.

If we include n Dp-branes (and their images), there are two possible actions on the open string Chan-Paton factors [9],

$$\lambda\to M^{-1}\lambda^T M\,,\quad M=\mathbb{I}_{2n\times 2n}\ \text{ or }\ \begin{bmatrix}0 & i\mathbb{I}_{n\times n}\\ -i\mathbb{I}_{n\times n} & 0\end{bmatrix}(\equiv\mathbb{J})\,. \tag{2.3}$$

In the first case the resulting gauge group is $\mathrm{SO}(2n)$, and in the second case $\mathrm{U\,Sp}(2n)$. The corresponding Möbius strip amplitudes, and therefore also the vacuum $\mathbb{R}\mathrm{P}^2$ amplitudes, have the opposite sign, so we see that there are actually two types of Op-planes. These are denoted Op^- and Op^+ for the $\mathrm{SO}(2n)$ and $\mathrm{U\,Sp}(2n)$ cases, respectively. Comparing the Möbius strip amplitude with the cylinder amplitude, one finds that the RR charge and tension of the Op-planes (as compared with those of the Dp-branes) is given by -2^{p-5} for Op^-, and $+2^{p-5}$ for Op^+ (hence the sign in the notation).

2.2 Cohomology torsion variants

Another way to see that there are (at least) two variants (at least for $p\leq 6$) is to note that the orientifold background admits a discrete degree of freedom corresponding to the holonomy of the B field [1],

$$b=\int_{\mathbb{R}\mathrm{P}^2}\frac{B}{2\pi}=0 \qquad \text{or} \qquad \frac{1}{2}\,, \tag{2.4}$$

where $\mathbb{R}\mathrm{P}^2\subset\mathbb{R}\mathrm{P}^{8-p}$ which surrounds the Op-plane. A non-trivial holonomy contributes $e^{2\pi i b}=-1$ in the $\mathbb{R}\mathrm{P}^2$ amplitude, and therefore exchanges the Op^- and Op^+ planes.[1] Since B is odd under the projection, which maps opposite points in the

[1]It isn't obvious at this stage which variant should be identified with trivial holonomy, and which with non-trivial holonomy. This will be clarified in K-theory.

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double cover of $\mathbb{RP}^{8-p}$, the holonomy is an element of $H^2(\mathbb{RP}^{8-p}, \widetilde{\mathrm{U}(1)})$, where $\widetilde{\mathrm{U}(1)}$ is the twisted U(1) bundle. The elements of $H^2(\mathbb{RP}^{8-p}, \widetilde{\mathrm{U}(1)})$ are in turn classified topologically by the characteristic class

$$[H] = [dB] \in H^3(\mathbb{RP}^{8-p}, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2\,, \tag{2.5}$$

where $\widetilde{\mathbb{Z}}$ is the twisted $\mathbb{Z}$ bundle. The choice of torsion coincides with the holonomy b.

RR fields can also have a discrete holonomy, which leads to additional variants of the orientifold plane. For the Op-plane with $p \leq 5$ the relevant holonomy is

$$c = \int_{\mathbb{RP}^{5-p}} \frac{C_{5-p}}{2\pi} = 0 \qquad \text{or} \qquad \frac{1}{2}\,, \tag{2.6}$$

and is identified with a torsion G_{6-p}-field,

$$[G_{6-p}] \in \left\{ \begin{array}{ll} H^{6-p}(\mathbb{RP}^{8-p}, \mathbb{Z}) & \text{for } \ p = 0, 2, 4 \\ H^{6-p}(\mathbb{RP}^{8-p}, \widetilde{\mathbb{Z}}) & \text{for } \ p = 1, 3, 5 \end{array} \right\} = \mathbb{Z}_2\,. \tag{2.7}$$

The choice of a non-trivial c is denoted $\widetilde{\mathrm{O}p}^{\pm}$. Unlike the previous case, the charges of these variants cannot be computed in perturbation theory, since they involve a RR background. Instead, one must appeal to arguments involving strong coupling. For example, the O3-plane has four variants corresponding to H and G_3 torsion, both of which take values in $H^3(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$. S-duality in type-IIB string theory exchanges H and G_3, so O3$^-$ and $\widetilde{\mathrm{O3}}^+$ are invariant, and O3$^+$ is interchanged with $\widetilde{\mathrm{O3}}^-$ [1]. If we include D3-branes, string S-duality becomes Olive-Montonen duality in the world-volume $N = 4$ supersymmetric gauge theory. The SO($2n$) theory corresponding to O3$^-$ is invariant under this duality, but the U Sp($2n$) theory of O3$^+$ is related to an SO($2n+1$) theory [10]. We can therefore interpret the $\widetilde{\mathrm{O3}}^-$ plane as an O3$^-$ plane with an additional fractional (1/2) D3-brane. The fourth variant, $\widetilde{\mathrm{O3}}^+$, corresponds to turning on H torsion on $\widetilde{\mathrm{O3}}^-$. This exchanges the orthogonal gauge group with a symplectic one, U Sp($2n$). The theory is perturbatively the same as for O3$^+$, but the BPS dyon spectrum is different [2]. This U Sp($2n$) theory is self-dual.

There are analogous strong coupling arguments for type-IIA O2 planes [11] and O4 planes [12]–[14] using M-theory. However, it is clear from T-duality that this pattern persists for all $p \leq 5$ (see table 1).

The case $p = 6$ is special, since $[G_0] \in H^0(\mathbb{RP}^{8-p}, \mathbb{Z}) = \mathbb{Z}$, rather than $\mathbb{Z}_2$. These classes correspond to the allowed (integral) values of the cosmological constant in massive type-IIA supergravity [15, 16]. It does not appear, at this stage, to label distinct O6-planes, but rather different IIA backgrounds in which an O6-plane can sit. One might therefore conclude that the O6-plane only has two variants, O6$^-$ and O6$^+$. However, we shall see in section 5 that there is another variant, which has the properties (namely charge and tension) of the $\widetilde{\mathrm{O}p}^-$-planes, and is therefore similarly denoted $\widetilde{\mathrm{O6}}^-$ (see also [17]).

For $p = 0$ and 1 there is another possible holonomy given by

$$c' = \int_{\mathbb{RP}^{1-p}} \frac{C_{1-p}}{2\pi} = 0 \qquad \text{or} \qquad \frac{1}{2}\,. \tag{2.8}$$

This is related to the torsion groups $H^1(\mathbb{RP}^7, \widetilde{\mathbb{Z}})$ and $H^2(\mathbb{RP}^8, \mathbb{Z})$. We denote the corresponding variants $\widehat{\mathrm{O1}}$ and $\widehat{\mathrm{O0}}$. When both RR torsions are turned on we get more variants, $\widehat{\widetilde{\mathrm{O1}}}$ and $\widehat{\widetilde{\mathrm{O0}}}$. The charges of these variants were not known. Using K-theory, we will be able to determine their charges. We will also see that some of these variants are removed when one interprets RR fields in K-theory. For the O2-plane, one could also consider a G_0 background, which takes values in $H^0(\mathbb{RP}^6, \mathbb{Z}) = \mathbb{Z}$. The situation is similar to the O6-plane case above, and will also be addressed in section 5.

2.3 Brane realization of torsion

Pure torsion fields can be realized physically by simple brane constructions [18, 19, 2]. Consider an Op-plane ($p \leq 5$) situated along the directions $\{x^1, \ldots, x^{p-1}, x^6\}$, and either an NS5-brane along $\{x^1, \ldots, x^5\}$ for the case of the H field, or a D(p+2)-brane along $\{x^1, \ldots, x^{p-1}, x^7, x^8, x^9\}$ for the case of the G_{6-p} field (figure 1).[2] In both cases the brane intersects the orientifold plane along $\{x^1, \ldots, x^{p-1}\}$, and therefore forms a "domain wall" on the Op-plane. The brane is "fractional" in the sense that it is its own image under the orientifold projection, and can therefore not move off the orientifold plane. It is therefore a source of 1/2 unit of flux in the reduced space, i.e.

$$\int_{\mathbb{S}^{3,1}} \frac{H}{2\pi} = \frac{1}{2} \tag{2.9}$$

for the NS5-brane, and

$$\int_{\mathbb{S}^{6-p,1}} \frac{G_{6-p}}{2\pi} = \frac{1}{2} \tag{2.10}$$

for the D(p+2)-brane.[3] Now remove two small disks at the intersection of the sphere with the orientifold plane, and apply Stoke's law to get

$$\Delta \int_{\mathbb{RP}^2} \frac{B}{2\pi} = \Delta \int_{\mathbb{RP}^{5-p}} \frac{C_{5-p}}{2\pi} = \frac{1}{2} \tag{2.11}$$

for the difference in the holonomy between the two sides. The field strengths H and G_{6-p} are trivial (as differential forms) at infinity, but the above result shows that there is relative torsion between the two pieces of the Op-plane. The NS5-brane interchanges the Op^- and Op^+ planes, and the D($p+2$)-brane interchanges the Op and $\widetilde{\mathrm{O}p}$ planes.

[2] A similar construction can be made for the additional RR torsion (c') in the case of $p = 0, 1$, using an intersection with a D6-brane and D7-brane, respectively.

[3] We use $\mathbb{S}^{l,m}$ to denote the sphere in the reduced space $\mathbb{R}^{l,m} \equiv \mathbb{R}^{l+m}/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ inverts l of the coordinates. In particular $\mathbb{S}^{l,0} \sim \mathbb{RP}^{l-1}$.

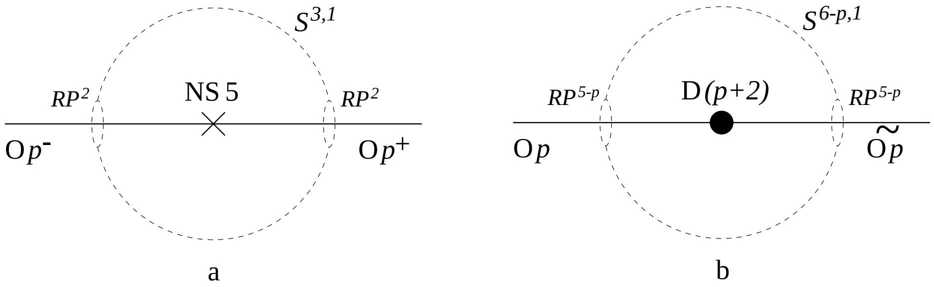

Figure 1: Brane realization of (a) H torsion, (b) G_{6-p} torsion.

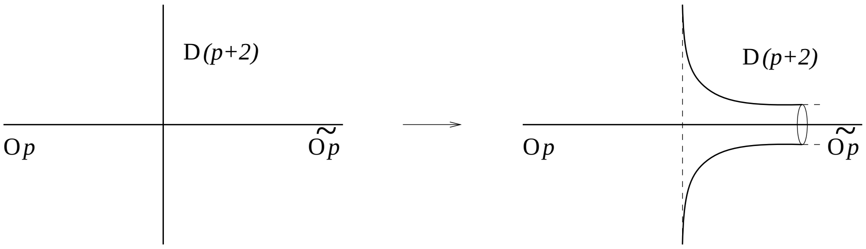

Figure 2: Deforming the intersecting brane to a wrapped brane.

For two coincident branes the flux is 1, but they are now free to move off the orientifold plane (in opposite directions). The torsion part of the field is therefore trivial, and the orientifold plane remains unchanged.

For $p = 6$ the D-brane picture (figure 1b) is special, in that G_0 is created by a D8-brane, which fills the entire transverse space. The D8-branes can therefore never move off the orientifold plane, and this is consistent with the fact that G_0 is integral rather than torsion.

We can try to use this picture to understand the relative charges of the $\widetilde{\mathrm{O}p}^{\pm}$ planes. Consider a "deformation" of the intersecting D-brane configuration, in which the D$(p+2)$-brane wraps $\mathbb{R}^p \times \mathbb{RP}^2$ at $x^6 \to \infty$, and "opens up" into a flat $\mathbb{R}^{p+2}$ at $x^6 = 0$ (figure 2). The $\mathbb{RP}^2$ is not a homology cycle in the bulk, so the wrapped part of the brane shrinks to zero size, giving the original intersection configuration. If we assume that the world-volume gauge field on the D$(p+2)$-brane satisfies [20, 17]

$$\int_{\mathbb{RP}^2} \frac{F}{2\pi} = \frac{1}{2} \qquad \mathrm{mod}\ \mathbb{Z}\,, \tag{2.12}$$

then the charge of the Op^- plane would shift by 1/2, due to the coupling $\int F \wedge C_{p+1}$. On the other hand the charge of Op^+ could only change by an integral amount due the additional coupling $\int B \wedge C_{p+1}$, since the B field also has 1/2 unit of flux on $\mathbb{RP}^2$. This would therefore explain the observed relative charges of these two variants. A result similar to (2.12) was obtained in [21] for orientable manifolds for which w_2 is non-trivial. However $\mathbb{RP}^2$ is non-orientable, so the question remains open.

3. Orientifolds in K-theory

In ordinary (non-orientifold) type-IIA and -IIB backgrounds RR fluxes take values in $K(X)$ and $K^{-1}(X)$, respectively [7]. This complements the fact that RR charges, or D-branes, are classified by $K^{-1}(X)$ and $K(X)$, respectively.[4] One way to think about this, which will be useful in generalizing to orientifolds, is to divide the space into two regions using a domain wall made of D8-branes and anti-D8-branes in IIA, and non-BPS D8-branes in IIB. Differences in the values of the RR fluxes between the two regions are then in one-to-one correspondence with the RR charges carried by the D8-brane system, and are therefore classified by $K(X)$ in IIA, and by $K^{-1}(X)$ in IIB. The groups are exchanged relative to the D-brane classification since there we start with a D9-$\overline{\text{D9}}$ system in IIB, and non-BPS D9-branes in IIA.

3.1 Real K-theory

The orientifold backgrounds of interest are defined by a $\mathbb{Z}_2$ projection on closed strings that includes world-sheet parity Ω, and an action on the open string Chan-Paton factors that squares to either 1 or -1. In K-theoretic terms, one is given an involution τ on the space X, and an antilinear involution on the bundles that commutes with τ. This defines the real and quaternionic K-theory groups $KR(X)$ and $KH(X)$, depending on whether the involution on the bundles squares to 1 or -1 [22]. Higher groups can then be defined by (assuming compact support)

$$KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q})\,, \tag{3.1}$$

where the involution acts on p coordinates in $\mathbb{R}^{p,q}$, and similarly for KH. These satisfy the periodicity conditions

$$KR^{p,q}(X) = KR^{p+1,q+1}(X) \tag{3.2}$$

$$KR^{p,q}(X) = KR^{p+8,q}(X)\,, \tag{3.3}$$

and similarly for KH. The first implies that $KR^{p,q}$ depends only on the difference $p-q$, so we denote $KR^{p,q}(X) = KR^{p-q}(X)$. In addition KR and KH are related by

$$KR^{-n}(X) = KH^{-n+4}(X)\,. \tag{3.4}$$

Another variant of real K-theory can be defined by considering an involution that also exchanges the two bundles E and F. Physically this corresponds to the inclusion of the operator $(-1)^{F_L}$ (e.g. for $p = 3 \bmod 4$ in (2.1)), which exchanges the branes and antibranes. The corresponding groups are denoted $KR_\pm(X)$ and $KH_\pm(X)$, and

[4] If X is non-compact, then D-branes are actually classified by K-theory with compact support, $K^{-1}_{\text{cpct}}(X)$ and $K_{\text{cpct}}(X)$, or more precisely by the kernel of the natural maps from these groups to the ones without compact support.

higher versions are defined precisely as in (3.1). These groups are related to ordinary real K-theory as follows,

$$KR_{\pm}(X) = KR(X \times \mathbb{R}^{0,2}) = KR^{-2}(X)\,, \tag{3.5}$$

and similarly for $KH_{\pm}$.[5]

3.2 RR fields in orientifold backgrounds

To classify RR fields in the background of an orientifold p-plane we divide the transverse space $\mathbb{R}^{9-p}$ by wrapping D8-branes on the transverse sphere $\mathbb{S}^{8-p}$. In IIA we again consider both D8-branes and anti-D8-branes, and in IIB we use the non-BPS D8-branes. The orientifold projection (2.1) acts freely on the sphere, giving $\mathbb{RP}^{8-p}$, and the corresponding K-groups are obtained by keeping track of how the projection acts on the D8-branes and their bundles.

For example, the O0 projection $\mathbb{R}^9/\mathcal{I}_9\,\Omega$ in IIA maps a D8-brane wrapping $\mathbb{S}^8$ with a given orientation to a D8-brane wrapping it with the opposite orientation. It therefore exchanges wrapped D8-branes with wrapped anti-D8-branes, and vice-versa. The group is therefore either $KR_{\pm}(\mathbb{S}^{9,0})$ or $KH_{\pm}(\mathbb{S}^{9,0})$, depending on whether the action on the Chan-Paton bundle squares to 1 or -1. Following the general arguments of [9], we associate the former to the O0^--plane, and the latter to the O0^+-plane.

The usual 8-fold periodicity of orientifolds (and real K-theory) implies that for the O8^- and O8^+ planes the groups are also $KR_{\pm}$ and $KH_{\pm}$, respectively. This is somewhat counterintuitive, in that a D8-brane gets mapped to an image D8-brane and not an anti-D8-brane. However recall that the D8-branes wrap the transverse sphere, which in this case is just two points on either side of the O8-plane. This means that the orientation of the "wrapped" D8-brane is reversed on the other side. The projection maps the D8-brane to a D8-brane of the same orientation on the other side, so it looks like a "wrapped" anti-D8-brane.

The appropriate groups for the other orientifold planes can be similarly determined. The results are summarized in table 2.

Using (3.2)–(3.5) we can re-express these groups as $KR^{p-10}(\mathbb{S}^{9-p,0})$ for $\text{O}p^-$, and $KH^{p-10}(\mathbb{S}^{9-p,0}) = KR^{p-6}(\mathbb{S}^{9-p,0})$ for $\text{O}p^+$. Note that, as expected, the groups which classify RR fields are shifted by one relative to the groups which classify RR charges, KR^{p-9} and KH^{p-9} [23]–[25]. For example, D-branes in an O8 background (type IA)

[5] This relation is the analog in real K-theory of Hopkins' relation in complex equivariant K-theory, $K_{\pm}(X) = K_{\mathbb{Z}_2}(X \times \mathbb{R}^{1,1})$. The latter can be understood as the failure of ordinary Bott periodicity when one tries to construct the Dirac index map: $K_{\mathbb{Z}_2}(X \times \mathbb{R}^{1,1}) \to K_{\mathbb{Z}_2}(X)$. Orientation reversal on the $\mathbb{R}^2$ exchanges the two spinor chiralities, and therefore gives $K_{\pm}(X)$ rather than $K_{\mathbb{Z}_2}(X)$. In the real case it's exactly the opposite, since complex conjugation also exchanges the two chiralities. Bott periodicity works for $\mathbb{R}^{1,1}$ (3.2), and fails for $\mathbb{R}^{0,2}$ and $\mathbb{R}^{2,0}$. The former gives $KR_{\pm}(X)$, and the latter $KH_{\pm}(X)$.

Op^-	K-group
$O0^-$	$KR_\pm(S^{9,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O1^-$	$KR^{-1}(S^{8,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O2^-$	$KR(S^{7,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O3^-$	$KH_\pm^{-1}(S^{6,0}) = \mathbb{Z}$
$O4^-$	$KH_\pm(S^{5,0}) = \mathbb{Z}$
$O5^-$	$KH^{-1}(S^{4,0}) = \mathbb{Z}$
$O6^-$	$KH(S^{3,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O7^-$	$KR_\pm^{-1}(S^{2,0}) = \mathbb{Z}$
$O8^-$	$KR_\pm(S^{1,0}) = \mathbb{Z}$

Op^+	K-group
$O0^+$	$KH_\pm(S^{9,0}) = \mathbb{Z}$
$O1^+$	$KH^{-1}(S^{8,0}) = \mathbb{Z}$
$O2^+$	$KH(S^{7,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O3^+$	$KR_\pm^{-1}(S^{6,0}) = \mathbb{Z}$
$O4^+$	$KR_\pm(S^{5,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O5^+$	$KR^{-1}(S^{4,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$
$O6^+$	$KR(S^{3,0}) = \mathbb{Z} \oplus \mathbb{Z}$
$O7^+$	$KH_\pm^{-1}(S^{2,0}) = \mathbb{Z}$
$O8^+$	$KH_\pm(S^{1,0}) = \mathbb{Z}$

Table 2: Orientifold K-theory groups for RR fields.

are classified by KR^{-1}, whereas RR fields take values in $KR_\pm = KR^{-2}$. To compute the groups we used the following decomposition isomorphism due to Atiyah [22]

$$KR^{-n}(\mathbb{S}^{m,0}) = KR^{m-n+1}(pt.) \oplus KR^{-n}(pt.) \qquad (m \geq 3)\,, \tag{3.6}$$

and our knowlege of the real K-groups of a point,

$$KR^{-m}(pt.) = \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0\} \;\bmod\; 8\,. \tag{3.7}$$

3.3 Comparing with cohomology

The above results agree with cohomology in some cases, and differ in other cases. The free ($\mathbb{Z}$) part of K-theory always agrees with cohomology. This corresponds to the RR flux of the orientifold p-plane itself, i.e. G_{8-p}. For $p = 2, 6$ there is an additional free part corresponding to G_0, which also agrees with cohomology. The difference between K-theory and cohomology can only be in the torsion subgroup.[6]

For example, the relevant cohomologies for orientifold 5-planes are

$$H^3(\mathbb{RP}^3, \mathbb{Z}) \oplus H^1(\mathbb{RP}^3, \widetilde{\mathbb{Z}}) = \mathbb{Z} \oplus \mathbb{Z}_2\,. \tag{3.8}$$

The first term corresponds to the integral G_3 flux of the O5-plane, and the second is G_1 torsion, which gives the $\widetilde{\mathrm{O}5}$ variants. Let us once again stress that these cohomologies do not incorporate the fractional shift in the G_3 flux of $\widetilde{\mathrm{O}5}^-$ relative to $\mathrm{O}5^-$. In K-theory we find

$$\begin{aligned} KH^{-1}(\mathbb{S}^{4,0}) &= \mathbb{Z} && \text{for O}5^- \\ KR^{-1}(\mathbb{S}^{4,0}) &= \mathbb{Z} \oplus \mathbb{Z}_2 && \text{for O}5^+\,. \end{aligned} \tag{3.9}$$

How should we interpret the absence of the torsion subgroup in the first case? Clearly we need to understand better how to relate (or "lift") cohomology to K-theory.

[6]This is a consequence of the Chern isomorphism, e.g. $K(X) \times \mathbb{Q} \simeq H^{even}(X, \mathbb{Q})$. In the real case the cohomologies will be twisted or normal, according to the action of the orientifold on the corresponding RR field (2.2).

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There is in fact a systematic algorithm which does exactly this, known as the Atiyah-Hirzebruch spectral sequence (AHSS) [26].[7] The basic idea of the AHSS is to compute $K(X)$ using a sequence of successive approximations, starting with integral cohomology $H^*(X,\mathbb{Z})$. Each successive step is given by the cohomology of a differential d_r that acts on classes in the previous step. Thus classes which are not d_r-closed are removed in the new approximation, and classes which are d_r-exact are trivial. For example, d_1 is just the usual co-boundary operator acting on co-chains (which are the zero'th step), so the first step is ordinary de-Rahm cohomology. Higher differentials, if they are non-trivial, will refine the approximation by removing some cohomology classes, and trivializing others. The first non-trivial higher differential in the complex case is given by $d_3 = Sq^3 + H$, where Sq^3 is the Steenrod square [27]. The only other differential which may be non-trivial in ten dimensions is d_5.

The AHSS can be extended to real K-theory as well. In the appendix we describe how to do this in the case of a freely acting involution, such as the ones appearing in table 2. In this case the AHSS follows a similar pattern to the complex case, with the modification that the cohomologies appearing in the first approximation can be twisted or normal, in accordance with the orientifold action on the RR fields (2.2). Correspondingly, the differential d_3 maps between ordinary and twisted cohomologies, and must therefore itself be twisted. The precise form of d_3 is not known, but it can presumably be expressed in the form $d_3 = \widetilde{Sq}^3 + H$, where $\widetilde{Sq}^3$ is a twisted version of the Steenrod square. Here the second term is torsion, since $[H] \in H^3(\mathbb{RP}^{8-p},\widetilde{\mathbb{Z}}) = \mathbb{Z}_2$. As we have seen before, this provides the difference between the Op^--plane and the Op^+-plane. We shall assume that the first term is trivial in both cases, and justify this assumption in hindsight. We shall also see that d_5 (which is untwisted) is trivial in all cases.

Actually, what one really computes in the AHSS is an "associated graded complex" of $K(X)$ (or any of its variants), denoted $\mathrm{Gr}K(X)$. This is defined in terms of a filtration of $K(X)$, $K_n(X) \subseteq K_{n-1}(X) \subseteq \cdots \subseteq K_0(X) = K(X)$, as

$$\mathrm{Gr}K(X) = \oplus_p K_p(X)/K_{p+1}(X)\,, \tag{3.10}$$

where $K_p(X)$ is the subgroup of classes in $K(X)$ which are trivial on the $(p-1)$-tree of X. The successive ratios appearing in (3.10) are precisely the objects one computes by the AHSS. In particular, in the first order approximation

$$K_p(X)/K_{p+1}(X) = \begin{cases} H^p(X,\mathbb{Z}) & \text{for } p \text{ even} \\ 0 & \text{for } p \text{ odd}\,. \end{cases} \tag{3.11}$$

To get the actual subgroups $K_p(X)$, and in particular $K(X)$ itself, requires solving extension problems of the form

$$0 \to K_{p+1}(X) \to K_p(X) \to K_p(X)/K_{p+1}(X) \to 0\,. \tag{3.12}$$

[7] Introductory reviews of the AHSS for complex K-theory can be found in the appendix, as well as in the appendices of [27] and [28].

If the above exact sequence splits, i.e. if it is exact both ways, the extension is said to be trivial, and $K_p(X) = K_{p+1}(X) \oplus K_p(X)/K_{p+1}(X)$. If all the extensions are trivial we deduce recursively that $K(X) = \mathrm{Gr}K(X)$. In other words, K-theory is isomorphic to the ring of cohomologies that survive the higher differentials of the AHSS. On the other hand, if some of the extensions are non-trivial $K(X)$ will differ from its graded complex. In particular, the additive structure in K-theory will be different from that of the cohomology classes. Physically this means that RR fields of different degree become correlated.

Consider for example the space $X = \mathbb{R}\mathrm{P}^5$, for which

$$\begin{aligned} \mathrm{Gr}K(\mathbb{R}\mathrm{P}^5) &= H^0(\mathbb{R}\mathrm{P}^5) \oplus H^2(\mathbb{R}\mathrm{P}^5) \oplus H^4(\mathbb{R}\mathrm{P}^5) \\ &= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \,. \end{aligned} \tag{3.13}$$

The relevant extension is given by

$$\begin{array}{ccccccc} 0 \to & K_3(\mathbb{R}\mathrm{P}^5) & \to & K_2(\mathbb{R}\mathrm{P}^5) & \to & K_2(\mathbb{R}\mathrm{P}^5)/K_3(\mathbb{R}\mathrm{P}^5) & \to 0 \\ & \| & & \| & & \| & \\ & H^4 = \mathbb{Z}_2 & & K_1(\mathbb{R}\mathrm{P}^5) & & H^2 = \mathbb{Z}_2 & \end{array} \,. \tag{3.14}$$

On the other hand it is known (see for example the appendix of [28]) that

$$K_1(\mathbb{R}\mathrm{P}^5) = \widetilde{K}(\mathbb{R}\mathrm{P}^5) = \mathbb{Z}_4 \,, \tag{3.15}$$

so this extension is non-trivial. The RR 2-torsion and 4-torsion appearing in $H^2(\mathbb{R}\mathrm{P}^5)$ and $H^4(\mathbb{R}\mathrm{P}^5)$ become correlated in K-theory; twice the generator of the former corresponds to the generator of the latter.

In practical terms, one often needs additional information to determine whether an extension is trivial or not, which precludes the AHSS as a useful method for computing $K(X)$ (as opposed to $\mathrm{Gr}K(X)$). However it will be extremely useful in our case, since we already know the precise groups (table 2). We are therefore able to determine all the extensions (3.12) unambiguously, and thereby compare the K-theory data with cohomology.

4. Applications

Having elaborated the K-theory machinery needed to classify RR fields surrounding orientifolds, we would now like to demonstrate three important implications of the fact that that RR fields are valued in K-theory rather than cohomology.

4.1 Charge quantization

As was previously stated, there are two separate issues related to the question of proper RR charge quantization. The first has to do with the absolute fractional charges (of the Op-planes with $p \leq 4$, for example), and the second with the relative charge induced by RR torsion (1/2 for Op^-, 0 for Op^+). The first issue encompasses the second, and should in principle be resolved by an analog of the relation [7]

$$\frac{G(x)}{2\pi} = \sqrt{\widehat{A}}\,\mathrm{ch}(x)\,, \qquad x \in K(X) \tag{4.1}$$

for real K-theory. Unfortunately the generalization is not known, so we will concentrate on the issue of the relative charge. The main idea, as alluded to in the previous section, is that in K-theory RR fields of different degree become correlated via non-trivial extensions.

Note first that for the O8 and O7-planes K-theory is in complete agreement with cohomology. There is only a single RR flux, and therefore no possibility for a non-trivial extension. There are only two variants of these planes, Op^- and Op^+. In particular the O8-plane (O7-plane) cannot support a fractional D8-brane (D7-brane). This is consistent with other arguments which rule out the possibility of a gauge group SO$(2n+1)$ in these cases [17].

The O6-plane background supports two RR fluxes, G_2 and G_0, but both are integral. Since the differential d_3 (and d_5) is obviously trivial in this case, the AHSS terminates at the first approximation, i.e.

$$\begin{aligned} K_0/K_1 &= H^0(\mathbb{RP}^2,\mathbb{Z}) = \mathbb{Z} \\ K_1/K_2 &= 0 \\ K_2 &= H^2(\mathbb{RP}^2,\widetilde{\mathbb{Z}}) = \mathbb{Z}\,. \end{aligned} \tag{4.2}$$

Using $K_0 = KH(\mathbb{S}^{3,0})$ and $KR(\mathbb{S}^{3,0})$ for O6^- and O6^+, respectively, we can determine the filter groups uniquely,

$$\begin{aligned} K_0 &= \mathbb{Z}\oplus\mathbb{Z} \\ K_1 &= K_2 = \mathbb{Z}\,. \end{aligned} \tag{4.3}$$

However the extension given by

$$\begin{array}{ccccccc} 0 \to & K_1 & \to & K_0 & \to & K_0/K_1 & \to 0 \\ & \| & & \| & & \| & \\ & H^2 = \mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & H^0 = \mathbb{Z} & \end{array} \tag{4.4}$$

does not have a unique solution, so one cannot determine in this way whether and how the two RR fluxes are correlated. We will return to the O6-plane in section 5.

The first case where this procedure gives a non-trivial result is the O5-plane. The cohomologies and K-theory for the O5-planes were given in (3.8) and (3.9). The only possible non-triviality of d_3 could be in the map

$$H^0(\mathbb{RP}^3,\widetilde{\mathbb{Z}}) \xrightarrow{d_3} H^3(\mathbb{RP}^3,\mathbb{Z})\,. \tag{4.5}$$

However $H^0(\mathbb{RP}^3, \widetilde{\mathbb{Z}}) = 0$, so d_3 (and d_5) is trivial for both O5$^-$ and O5$^+$. The AHSS again terminates at cohomology, and we find

$$\begin{aligned} K_0 = K_1 &= \begin{cases} KH^{-1}(\mathbb{S}^{4,0}) = \mathbb{Z} & \text{for O5}^- \\ KR^{-1}(\mathbb{S}^{4,0}) = \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for O5}^+ \end{cases} \\ K_2 = K_3 &= H^3(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z} \qquad\qquad \text{for both} . \end{aligned} \tag{4.6}$$

This time we find that the extension given by

$$\begin{array}{ccccccc} 0 \to & K_2 & \to K_1 \to & K_1/K_2 & \to 0 \\ & \| & & \| & \\ & H^3 = \mathbb{Z} & & H^1 = \mathbb{Z}_2 & \end{array} \tag{4.7}$$

is trivial for O5$^+$, but non-trivial for O5$^-$. The generator of H^3 maps to the generator of (the free part of) K_1 for O5$^+$, but to twice the generator for O5$^-$. In the latter case the generator of K_1 maps to non-trivial torsion in H^1. We therefore see that G_1 torsion produces a half-integral shift in the flux of G_3 for the O5$^-$ plane, and has no effect on the charge of the O5$^+$ plane.

The story repeats along similar lines for the lower Op^- planes. The relevant cohomologies are $H^{8-p}(\mathbb{RP}^{8-p}) = \mathbb{Z}$ and $H^{6-p}(\mathbb{RP}^{8-p}) = \mathbb{Z}_2$, which are appropriately twisted or untwisted. The differential d_3 is trivial since $H = 0$ (and we assume $\widetilde{Sq^3} = 0$). Possible non-trivialities of d_5 could arise from the (twist preserving) maps

$$\begin{aligned} H^{3-p}(\mathbb{RP}^{8-p}) &\longrightarrow H^{8-p}(\mathbb{RP}^{8-p}) \\ H^{1-p}(\mathbb{RP}^{8-p}) &\longrightarrow H^{6-p}(\mathbb{RP}^{8-p}) \\ H^{6-p}(\mathbb{RP}^{8-p}) &\longrightarrow H^{11-p}(\mathbb{RP}^{8-p}) , \end{aligned} \tag{4.8}$$

however in all cases the maps are found to be trivial. The relevant extension is

$$\begin{array}{ccccccc} 0 \to & K_{7-p} & \to K_{6-p} \to & K_{6-p}/K_{7-p} & \to 0 \\ & \| & \| & \| & \\ & H^{8-p} = \mathbb{Z} & \mathbb{Z} & H^{6-p} = \mathbb{Z}_2 & \end{array} , \tag{4.9}$$

so the half-integral part of the G_{8-p} flux, i.e. the charge of the orientifold plane, is determined by the G_{6-p} torsion.

Actually, for $p \leq 2$ there is an additional RR cohomology H^{2-p}, i.e.

$$\begin{aligned} H^0(\mathbb{RP}^6, \mathbb{Z}) &= \mathbb{Z} && \text{for } p = 2 \\ H^1(\mathbb{RP}^7, \widetilde{\mathbb{Z}}) &= \mathbb{Z}_2 && \text{for } p = 1 \\ H^2(\mathbb{RP}^8, \mathbb{Z}) &= \mathbb{Z}_2 && \text{for } p = 0 . \end{aligned} \tag{4.10}$$

As with the other cohomologies, these lift unobstructed to the graded complex; d_3 is trivial since $H = 0$ for Op^- planes, and d_5 is trivial since $H^{7-p}(\mathbb{RP}^{8-p})$ (appropriately normal or twisted) is trivial. There is also an additional extension to solve,

$$\begin{array}{c} 0 \to K_{3-p} \to K_{2-p} \to K_{2-p}/K_{3-p} \to 0 \\ \hphantom{0 \to K_{3-p} \to K_{2-p} \to{}} \| \hphantom{{}\to 0} \\ \hphantom{0 \to K_{3-p} \to K_{2-p} \to{}} H^{2-p} \hphantom{{}\to 0} \end{array} . \tag{4.11}$$

One can show that in all three cases $K_{3-p} = \mathbb{Z}$, and $K_{2-p} = K_0$, which is $\mathbb{Z} \oplus \mathbb{Z}$ for $p = 2$, and $\mathbb{Z} \oplus \mathbb{Z}_2$ for $p = 0, 1$. The $\mathrm{O}2^-$ case is similar to the $\mathrm{O}6^-$ case above; there is no unique solution to the extension, so one must use a different method to determine the effect of G_0. We will return to this problem in section 5. For $\mathrm{O}1^-$ and $\mathrm{O}0^-$ the extension is trivial, so G_{2-p} torsion does not shift their charge.[8] The variants we called $\widehat{\mathrm{O}1}^-$ and $\widehat{\mathrm{O}0}^-$ therefore have the *same* charge (and tension) as the original torsion-free planes. It would be interesting to investigate whether (and how) the gauge theories on D1-branes and D0-branes in the presence of these exotic orientifold variants differ from the ordinary $\mathrm{SO}(2n)$ theories on $\mathrm{O}1^-$ and $\mathrm{O}0^-$.

The story becomes even more interesting for the lower $\mathrm{O}p^+$ planes, as we will see next.

4.2 Equivalent orientifolds

$\mathrm{O}p^+$ planes correspond to $\mathrm{O}p^-$ planes with non-trivial H-torsion. This will modify the AHSS for $p \leq 3$ through the differential d_3. In particular, for the $\mathrm{O}3^+$ plane

$$d_3 : H^0(\mathbb{RP}^5, \mathbb{Z}) \xrightarrow{\cup H} H^3(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) \tag{4.12}$$

is a surjective map from $\mathbb{Z}$ to $\mathbb{Z}_2$. G_3 torsion therefore lifts to a trivial class, and the graded complex is simply $\mathrm{Gr}KR_{\pm}^{-1}(\mathbb{S}^{6,0}) = \mathbb{Z}$. As with the $\mathrm{O}5^+$ plane, the extensions are trivial, so the charge doesn't shift. However, the new feature here is that, since the two G_3 fluxes are equivalent in K-theory, the two variants $\mathrm{O}3^+$ and $\widetilde{\mathrm{O}3}^+$ are equivalent as well. Physically this can be understood as follows. Equation (4.12) implies the existence of a constant 0-form ω_0, such that

$$G_3 \;=\; H \cup \omega_0 \,, \tag{4.13}$$

and the two $\mathrm{O}3^+$ variants are related by a shift $\omega_0 \to \omega_0 + 1$. On the other hand, the NSNS and RR 3-torsions of the two variants are related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation in type-IIB string theory given by [1, 2]

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \tag{4.14}$$

which also shifts the axion $C_0 \to C_0 + 1$. We therefore identify ω_0 with C_0.

[8] This is also consistent with the fact that the branes which create this torsion, i.e. a D6-brane for $\mathrm{O}0^-$ and a D7-brane for $\mathrm{O}1^-$, cannot support the end of a D-string.

Note that this transformation is global; in a background with several $O3^+$ planes, it changes all of them into $\widetilde{O3}^+$ planes. In particular, a configuration with two $O3^+$ planes at opposite points on a circle is not equivalent to a configuration with one $O3^+$ and one $\widetilde{O3}^+$. This is also consistent with the fact that these configurations are related by T-duality to an $O4^+$ plane and an $\widetilde{O4}^+$ plane, respectively (figure 3). The latter are inequivalent, since they differ by a (G_2) torsion class which is not trivial in K-theory.

For $p \leq 2$ we similarly find that G_{6-p} torsion can be written as

$$G_{6-p} = H \cup \omega_{3-p}\,, \tag{4.15}$$

where $\omega_{3-p} \in H^{3-p}(\mathbb{RP}^{8-p}) = \mathbb{Z}_2$ (appropriately normal or twisted), so Op^+ and $\widetilde{Op}^+$ are equivalent in K-theory. For $p = 2$ there is again an interesting physical picture. Consider the lift of $O2^+$ to M-theory [11, 2]. This corresponds to an $OM2^+$ plane and and $OM2^-$ plane located at opposite points on the circle.[9] The $\widetilde{O2}^+$ plane is then simply gotten by exchanging the positions of the two OM2 planes, which is equivalent to the shift $x^{10} \to x^{10} \pm \pi$. On the other hand, the M-theory shift $x^{10} \to x^{10} + \xi$ $(-\pi \leq \xi \leq \pi)$ in flat space reduces to a U(1) gauge transformation in type-IIA string theory,

$$\delta C_1 = d\xi\ ,\ \delta C_3 = \frac{B}{2\pi} \wedge d\xi\,. \tag{4.16}$$

In the OM2 background we require $\xi(-x^i) = -\xi(x^i)$, where x^i $(i = 3, \ldots, 9)$ are the coordinates transverse to the $O2^+$-plane. In particular, at the location of the orientifold plane $\xi(0) = 0$ or π (mod2π) only, and this labels the discrete choice of the $O2^+$ variant. Comparing (4.16) with (4.15) we are led to identify ω_1 with flat RR 1-forms $C_1 \in H^1(\mathbb{RP}^6, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$.

In this case there are two possible non-trivial transformations for a pair of $O2^+$ planes at opposite points on a circle,

$$\xi_1(x^9) = \begin{cases} -\pi & \text{for} \quad -\pi < x^9 < 0 \\ \pi & \text{for} \quad 0 < x^9 < \pi \end{cases}\,, \qquad \xi_2(x^9) = x^9\,. \tag{4.17}$$

In the first case both are transformed into $\widetilde{O2}^+$ planes, but in the second case only one (at $x^9 = \pi$) is transformed. The configuration with one $O2^+$ and one $\widetilde{O2}^+$ is therefore equivalent to the one with two $O2^+$ planes. This is unlike the $O3^+$ case, since here the T-dual configurations are $\widetilde{O3}^+$ and $O3^+$, respectively (figure 3), which are equivalent.

In general, the third differential d_3 will not correspond to just the cup product with H, but the intuition remains. If a field strength G_n can be written as a derivative $d_m C_{n-m}$, then it can be trivialized by a gauge transformation, and does not truly index a new RR field configuration.

[9]Two $OM2^-$ planes correspond to an $O2^-$ plane, and two $OM2^+$ planes correspond to an $\widetilde{O2}^-$ plane

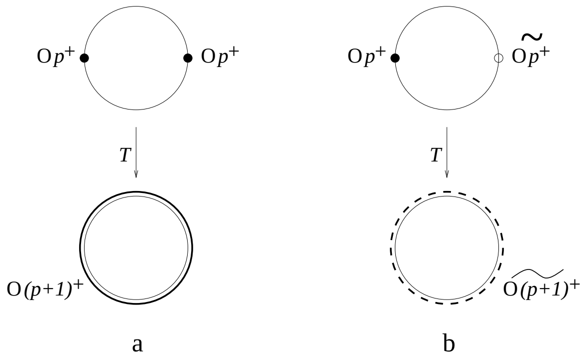

Figure 3: T-duality relates (a) a configuration of two $\mathrm{O}p^+$-planes at opposite points on a circle to a single $\mathrm{O}(p+1)^+$-plane wrapping the circle, and (b) a configuration of one $\mathrm{O}p^+$ and one $\widetilde{\mathrm{O}p}^+$ to a wrapped $\widetilde{\mathrm{O}(p+1)}^+$. The two configurations are equivalent for $p \leq 2$.

4.3 Unphysical orientifolds

For $\mathrm{O}p^+$ planes with $p \leq 2$ there is one more modification of the AHSS due to the differential d_3. The map

$$d_3 : H^{2-p}(\mathbb{R}\mathrm{P}^{8-p}) \xrightarrow{\cup H} H^{5-p}(\mathbb{R}\mathrm{P}^{8-p}) = \mathbb{Z}_2 \,, \tag{4.18}$$

where the cohomologies are appropriately normal or twisted (4.10), implies that G_{2-p} is not d_3-closed, but rather

$$d_3 G_{2-p} = \omega_{5-p} \in \mathbb{Z}_2 \,. \tag{4.19}$$

If ω_{5-p} is the non-trivial element of $\mathbb{Z}_2$ then the corresponding class in H^{2-p} cannot be lifted to K-theory. In particular, for the $\mathrm{O}0^+$ and $\mathrm{O}1^+$ planes G_{2-p} torsion is completely obstructed, which means that the variants $\widehat{\mathrm{O}0}^+$ and $\widehat{\mathrm{O}1}^+$ (as well as the composites $\widetilde{\widehat{\mathrm{O}0}}^+$ and $\widetilde{\widehat{\mathrm{O}1}}^+$) do not exist.

For the $\mathrm{O}2^+$ plane the above map is $\mathbb{Z} \to \mathbb{Z}_2$, which implies that odd values of $G_0/2\pi$ are not allowed. We will see in the next section that this can also be understood in terms of an anomaly in the three-dimensional gauge theory on D2-branes in an $\mathrm{O}2^+$ background with odd $G_0/2\pi$.

It would be interesting to find a physical interpretation for these obstructions, analogous to the interpretations we gave for the equivalences before.

5. The orientifold 6-plane

5.1 K-theory

In the previous section we used the AHSS to obtain new insight into how the cohomological classification of RR fluxes in orientifold backgrounds is modified by K-theory. In particular, we saw that independent RR fluxes may become correlated, certain

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RR fluxes become trivial, and others are obstructed. For the O6-planes the latter features are absent for dimensional reasons. On the other hand we were not able to determine the correlation of the two RR fluxes G_2 and G_0.

Luckily, representative gauge bundles for the relevant K-groups, $KH(\mathbb{S}^{3,0})$ and $KR(\mathbb{S}^{3,0})$, can easily be worked out. Start with a flat type-IIA background, and consider complex vector bundles over $\mathbb{S}^2$. These are classified by $K(\mathbb{S}^2) = K(pt.) \oplus \widetilde{K}(\mathbb{S}^2) = \mathbb{Z} \oplus \mathbb{Z}$, where the first term corresponds to the rank of the bundle, and the second term to its topological (ch_1) class. Physically these correspond to D8-branes (G_0 flux) and D6-branes (G_2 flux), respectively. A convenient basis of generators to work with is $x = (1, 0)$ and $y = (1, 1)$. The former corresponds to a constant rank 1 bundle v_0, and the latter to the basic rank 1 (monopole) line bundle with $ch_1 = 1$. This bundle can be constructed by assigning trivial bundles v_+ and v_- to the upper and lower hemishpheres, respectively, with a transition function along the equator,

$$v_+(\theta) = g(\theta) v_-(\theta) \,. \tag{5.1}$$

The transition function $g(\theta)$ is a map $\mathbb{S}^1 \to \mathrm{U}(n)$, which is classified by a winding number $w \in \pi_1(\mathrm{U}(n)) = \mathbb{Z}$. The basic line bundle has $w = ch_1 = 1$.

Now consider the action of the involution on the two generators. On the constant bundle the action is

$$v_0 \longrightarrow \bar{v}_0 \tag{5.2}$$

if the involution squares to 1, and

$$v_0 \longrightarrow \mathbb{J}\bar{v}_0 \tag{5.3}$$

if it squares to -1. Consequently x is a generator for $KR(\mathbb{S}^{3,0})$, whereas $2x$ is a generator for $KH(\mathbb{S}^{3,0})$. The action on the line bundle is given by

$$\begin{aligned} v_+(x) &\longrightarrow \bar{v}_-(-x) \\ v_-(x) &\longrightarrow \pm\bar{v}_+(-x) \,, \end{aligned} \tag{5.4}$$

where the upper sign corresponds to the KR projection and the lower sign to the KH projection. On the equator this becomes

$$\begin{aligned} v_+(\theta) &\longrightarrow \bar{v}_-(\theta + \pi) \\ v_-(\theta) &\longrightarrow \pm\bar{v}_+(\theta + \pi) \,. \end{aligned} \tag{5.5}$$

Compatibility with (5.1) requires

$$g(\theta + \pi) = \pm\bar{g}(\theta)^{-1} = \pm g(\theta)^T \,, \tag{5.6}$$

which implies that $w \in 2\mathbb{Z}$ for KR, and $w \in 2\mathbb{Z} + 1$ for KH. Thus y is a generator for $KH(\mathbb{S}^{3,0})$, whereas only $2y$ is a generator for $KR(\mathbb{S}^{3,0})$.

The results are summarized as follows:

$$\begin{array}{llll} \text{Covering Space}: & K(S^2) = \mathbb{Z}\oplus\mathbb{Z} & \text{generators } x=(1,0) & y=(1,1) \\ \text{O6}^-\text{plane}: & KH(S^{3,0}) = \mathbb{Z}\oplus\mathbb{Z} & \text{generators } 2x=(2,0) & y=(1,1) \\ \text{O6}^+\text{plane}: & KR(S^{3,0}) = \mathbb{Z}\oplus\mathbb{Z} & \text{generators } x=(1,0) & 2y=(2,2)\,. \end{array} \tag{5.7}$$

A unit G_2 flux, i.e. a bulk D6-brane, in the reduced space corresponds to the element $2y - 2x = (0, 2)$. We therefore see that half-odd-integral G_2 flux, i.e. a fractional D6-brane, is required for O6$^-$ when G_0 is odd, and forbidden when G_0 is even. In the former case the gauge group on D6-branes is SO$(2n + 1)$, and we denote the orientifold plane $\widetilde{\text{O6}}^-$. For O6$^+$ G_2 flux is always integral, and independent of G_0. The picture is similar to the one for the lower Op-planes, except that instead of the charge quantum being correlated with a RR torsion class (for Op^-), it is corellated with the parity of an integral class.

5.2 Anomalies

The above conclusion can also be arrived at by considering the world-volume theory on probe D-branes. It was shown in [17] that a Chern-Simons (CS) term which includes G_0 is crucial for cancellation of an anomaly with respect to large gauge transformations in the world volume theory of D2-branes in an $\widetilde{\text{O6}}^-$ background. We shall review the argument, and extend it to the other cases.

Consider first a flat type-IIA background. The CS terms on a Dp-brane (p even) are given by [29]

$$S^{flat}_{CS} = \int_{p+1} C \wedge \mathrm{tr} e^{F/2\pi} - \int_{p+1} \frac{G_0}{2\pi}\,\omega_{p+1}(A)\,, \tag{5.8}$$

where $\omega_{p+1}(A)$ is the Chern-Simons $(p + 1)$-form defined by

$$d\omega_{2n-1}(A) = 2\pi \cdot \frac{1}{n!\,(2\pi)^n}\mathrm{tr} F^n \tag{5.9}$$

up to exact forms. Here 'tr' denotes the trace in the fundamental representation of the gauge group. As is well known, the integral of a CS form is not gauge invariant, but rather shifts by a 2π multiple of the winding number of the gauge transformation. Namely, for an element $g \in$ U(n) with winding $w \in \pi_{p+1}(\mathrm{U}(n)) = \mathbb{Z}$, the gauge transformation $A \to g^{-1}Ag + g^{-1}dg$ induces

$$\int_{p+1} \omega_{p+1}(A) \to \int_{p+1} \omega_{p+1}(A) + 2\pi w\,. \tag{5.10}$$

Invariance of the path integral therefore requires $G_0/2\pi \in \mathbb{Z}$ [29].

In orientifold backgrounds the CS terms are given by

$$S_{CS} = \frac{1}{2}\int_{p+1} C \wedge \mathrm{tr} e^{F/2\pi} - \frac{1}{2}\int_{p+1} \frac{G_0}{2\pi}\,\omega_{p+1}(A)\,, \tag{5.11}$$

where now the traces are in the fundamental representation of $G = \mathrm{O}(n)$ or $\mathrm{U\,Sp}(2n)$, and the factor of $1/2$ is needed to correctly reproduce the action (5.8) when the D-branes move away from the orientifold plane. The cases of interest are $p = 2$ and 6, as these are the only cases (in ten dimensions) with integral winding, $\pi_3(G) = \pi_7(G) = \mathbb{Z}$. The integrals of the corresponding CS forms defined in (5.9) transform as follows

$$\delta \int \omega_3(A) = \begin{cases} 4\pi w & \text{for } \mathrm{O} \\ 2\pi w & \text{for } \mathrm{U\,Sp} \end{cases} \tag{5.12}$$

$$\delta \int \omega_7(A) = \begin{cases} 2\pi w & \text{for } \mathrm{O} \\ 4\pi w & \text{for } \mathrm{U\,Sp}\,, \end{cases} \tag{5.13}$$

where w is the appropriate winding number of the gauge transformation. The relative factors of 2 can be understood by considering $\mathrm{U\,Sp}$ and O as projections of U. The generator of $\pi_3(U)$ is associated with the instanton bundle on $\mathbb{S}^4$ (i.e. the generator of $\widetilde{K}(\mathbb{S}^4)$). Since this bundle is pseudoreal it survives the projection to $\mathrm{U\,Sp}$, but must be doubled to get the generator for O. For π_7 the situation is reversed, since the associated bundle on $\mathbb{S}^8$ is real.

There is an additional contribution to the CS action from the fermion determinant $\det(i\not{D})$ known as the parity anomaly. It was shown in [30] that a complex fermion in a representaion R of the gauge group contributes to the three-dimensional CS term as follows

$$S_{CS}^{\text{1-loop}} = \frac{1}{2}\frac{m}{|m|}\int \omega_3^R(A)\,, \tag{5.14}$$

where m is the mass of the fermion. A similar computation in seven-dimensional gauge theory gives

$$S_{CS}^{\text{1-loop}} = \frac{1}{2}\frac{m}{|m|}\int \omega_7^R(A)\,. \tag{5.15}$$

Consider first N D2-branes in the presence of an O6-plane and n D6-branes. The case of $\mathrm{O6}^-$ was considered in [17]. In this case the gauge group on the D2-branes is $\mathrm{U\,Sp}(2N)$, and the global symmetry corresponding to the D6-brane gauge group is $\mathrm{O}(2n)$ for $\mathrm{O6}^-$ and $\mathrm{O}(2n{+}1)$ for $\widetilde{\mathrm{O6}}^-$. The spectrum of (complex) fermions in the D2-brane world-volume theory consists of a state in the bi-fundamental representation $(\square,\square)$, as well as a pair of states in the symmetric representation of the gauge group $(\square\!\square,\mathbf{1})$, and a pair in the antisymmetric representation $(\begin{smallmatrix}\square\\ \square\end{smallmatrix},\mathbf{1})$. Invariance (mod 2π) of the sum of (5.11) and (5.14) therefore requires $G_0/2\pi$ to be even for $\mathrm{O6}^-$, and odd for $\widetilde{\mathrm{O6}}^-$. For $\mathrm{O6}^+$ the D2-brane gauge group is $O(2N)$ (or $O(2N+1)$), and the global symmetry is $\mathrm{U\,Sp}(2n)$. The fermion representations are the same as above, so in this case there is no parity anomaly. Furthermore, we see from (5.12) that $G_0/2\pi$ can be any integer. This is exactly what we obtained in K-thoery.

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The same result can also be obtained by considering the world-volume theory on the D6-branes. In this case there is a single complex fermion in the adjoint representation of the gauge group. The CS 7-forms defined in the adjoint representation are related to those defined in the fundamental representaion by

$$\int_{\Sigma_7} \omega_7^{\mathrm{adj}} = (m-8) \int_{\Sigma_7} \omega_7 \quad \text{for} \quad \mathrm{O}(m) \tag{5.16}$$

$$\int_{\Sigma_7} \omega_7^{\mathrm{adj}} = (2m+8) \int_{\Sigma_7} \omega_7 \quad \text{for} \quad \mathrm{U\,Sp}(2m)\,. \tag{5.17}$$

Together with (5.13) we see that, in the first case, invariance under a large gauge transformation requires $G_0/2\pi \in 2\mathbb{Z}$ for $\mathrm{O6}^-$ (m even), and $G_0/2\pi \in 2\mathbb{Z}+1$ for $\widetilde{\mathrm{O6}}^-$ (m odd). There is no such correlation for $\mathrm{O6}^+$, and $G_0/2\pi$ is an arbitrary integer.

The same approach can be applied to the O2-plane with G_0 flux, for which we only had partial K-theory results. In particular, we determined that $G_0/2\pi$ had to be even for $\mathrm{O2}^+$, but we made no statement about $\mathrm{O2}^-$. The world-volume theory on D2-branes in the background of an O2-plane contains an even number (four) of complex fermions in the adjoint representation in all cases, so there is no parity anomaly. Invariance under large gauge transformations therefore requires $G_0/2\pi \in 2\mathbb{Z}$ for $\mathrm{O2}^+$ (and $\widetilde{\mathrm{O2}}^+$), but allows any integer value for $\mathrm{O2}^-$ (and $\widetilde{\mathrm{O2}}^-$). This implies, in particular, that the G_6 charge quantum is not correlated with the value of G_0 (only with the torsion class of G_4, as we saw in the previous section).

5.3 Components of type-I vacuua

In many ways the orientifold 6-plane was the motivation for this investigation. Compactification of type-I string theory on $\mathbb{T}^3$, and T-duality, yields a background with eight $\mathrm{O6}^-$ planes and sixteen D6-branes. On the other hand it is known that the moduli space of type-I (or heterotic) vacua on $\mathbb{T}^3$ has additional disconnected components, with gauge groups of lower rank [31]–[35]. One of our original goals was to describe these different components in the T-dual picture, using D6-branes and orientifold 6-planes.

Some of the additional components can indeed be described in this way, by including both $\mathrm{O6}^-$ and $\mathrm{O6}^+$ planes, in much the same way as one describes the eight-dimensional theory without vector structure in terms of $\mathrm{O7}^-$ and $\mathrm{O7}^+$ planes [31]. In the fully compact case the seven-dimensional orientifold contains eight O6-planes, which are located at the vertices of a cube. Since each plane can be either $\mathrm{O6}^-$ or $\mathrm{O6}^+$, there are naively $2^8 = 256$ configurations. However not all of them are consistent, and many of them are related by the symmetry group of the 3-torus $\mathrm{SL}(3,\mathbb{Z})$.

Consider a configuration with N_- $\mathrm{O6}^-$ planes and N_+ $\mathrm{O6}^+$ planes. Supersymmetry requires $N_- \geq N_+$, since otherwise anti-D6-branes would be required for tadpole cancellation. Furthermore, consistency of the orientifold projection requires N_- and

N_+ to be even. For example, let us assume that the plane at the origin of the cube is O6$^+$, and the seven others are O6$^-$. If we denote the action of the projection on the CP factors by $\{g, g_1, g_2, g_3\}$, where g_i correspond to translations along the torus, then we have

$$\begin{aligned} g^T &= -g \\ (gg_i)^T &= +gg_i \qquad (i = 1,2,3) \\ (gg_ig_j)^T &= +gg_ig_j \qquad (i \neq j) \\ (gg_1g_2g_3)^T &= +gg_1g_2g_3 \,. \end{aligned} \tag{5.18}$$

From the first three relations we deduce that

$$\{g_i, g_j\} = 0 \quad (i \neq j)\,. \tag{5.19}$$

On the other hand this implies

$$\begin{aligned} (gg_1g_2g_3)^T &= -(gg_3g_2g_1)^T = -g_1^T g_2^T g_3^T g^T \\ &= -(-gg_1g^{-1})(-gg_2g^{-1}g)(-g_3g^{-1})(-g) \\ &= -gg_1g_2g_3\,, \end{aligned} \tag{5.20}$$

in contradiction with the last relation. The allowed values of (N_-, N_+) are thus $(8,0)$, $(6,2)$, and $(4,4)$ (figure 4). In the first case the configuration is unique and corresponds to the original rank 16 component (figure 4*a*). In the $(6,2)$ case the configuration is unique up to SL$(3,\mathbb{Z})$ transformations, and the rank is 8 (figure 4*b*). In the $(4,4)$ case there is no gauge group, and there are actually two SL$(3,\mathbb{Z})$ equivalence classes. One corresponds to configurations in which four O6-planes of the same type are co-planar (figure 4*c*), and the other to configurations where they are not co-planar (figure 4*d*).[10]

Given the previous results, we now understand that the above configurations require $G_0/2\pi$ to be even. In particular, they exist for $G_0 = 0$. For $G_0/2\pi$ odd on the other hand, we must replace O6$^-$ by $\widetilde{\text{O6}}^-$, and change the number of D6-branes appropriately to cancel the tadpole. This can only be done supersymmetrically in the $(8,0)$ and $(6,2)$ configurations, which yield new components with a rank 12 and a rank 5 gauge group, respectively. (In the $(4,4)$ configurations the total charge of the $\widetilde{\text{O6}}$-planes is $+2$, and can only be cancelled by anti-D6-branes.) Of course since $G_0 \neq 0$ in these cases, there is a non-vanishing vacuum energy, and supersymmetry is spontaneously broken.

[10]Since $M \in \mathrm{SL}(n,\mathbb{Z})$ satisfies $\det M = 1$, it follows that the Levi-Civita tensor $\epsilon_{i_1\cdots i_n}$, and therefore also the pseudo-scalar product of n vectors $\epsilon_{i_1\cdots i_n} v^{i_1}\cdots v^{i_n}$, is invariant under SL$(n,\mathbb{Z})$. For $n = 3$ this means that co-planarity is an SL$(3,\mathbb{Z})$-invariant property.

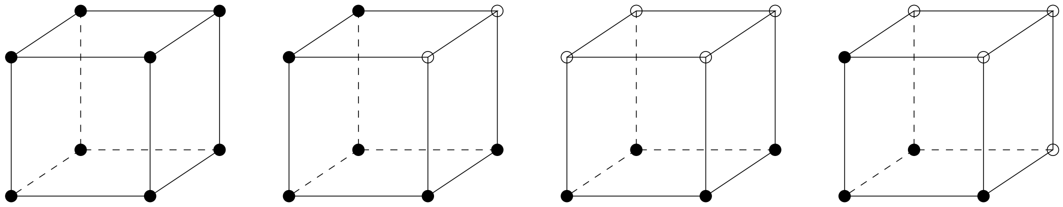

Figure 4: Consistent $\mathrm{SL}(3,\mathbb{Z})$-inequivalent configurations of O6-planes: (a) eight $\mathrm{O6}^-$ planes (rank 16), (b) six $\mathrm{O6}^-$ planes and two $\mathrm{O6}^+$ planes (rank 8), (c)+(d) two inequivalent configurations of four $\mathrm{O6}^-$ planes and four $\mathrm{O6}^+$ planes (rank 0).

6. Conclusions and open questions

In this paper we have demonstrated how K-theory generates very useful information on the nature of discrete RR G-fluxes in string theory, specifically for orientifold backgrounds. We saw that certain G-fluxes are correlated, that some become identified due to discrete transformations, and finally that some are disallowed outright. A very useful tool in this process was the Atiyah-Hirzebruch Spectral Sequence and our generalization of it. The AHSS ties together our understanding of separate RR-fields (even torsion ones) in terms of single K-classes.

The AHSS was particularly powerful for our set of examples because we already knew the relevant K-groups. However, use of the AHSS in general faces two obstacles. The first obstacle involves computing higher differentials in the sequence. We were able to circumvent this obstacle by using our exact knowledge of the K-groups. For complex K-theory, the d_3 differential has been explicitly worked out [26, 27]. For KR- and KH-theory, we did not work these out, although (along the lines of [21]) we can see that adding the torsion H-fields to d_3 brings us from one to the other. We would like to conclude by making some comments regarding the other obstacle: the matter of extensions.

One way to understand the extension problem is to realize that while the graded complex generated by the AHSS yields the elements of a K-group, it does not provide us with their additive structure. We propose that this additive structure can be extracted by thinking of the K-group elements on the asymptotic sphere as limiting values of generalized differentials in the bulk. For our purposes, a general differential form consists of a vector of differential forms of different ranks with a differential action which can increase the rank by more than one (for some ideas along these lines, we refer the reader to the ongoing project in [36, 8]).

The equation of motion for fields described by generalized differentials involve the generalized derivative (naively a combination of the d_1, d_3 and d_5 seen in the AHSS), which in general will not be zero but can also give a K-class corresponding to the source of the particular RR-fields. The novelty here is that a K-class corresponding to a p-form current J_p could be a source for any field strength G_m with $m \leq p-1$, instead of just for G_{p-1}. Consistency (commutativity) of the additive structure with the differential action will put useful restrictions on the additive structure of the generalized differential forms, which will in turn help determine the group structure of the K-group at infinity.

Another outstanding issue concerns the absolute charges of the orientifold. In this work we have only used K-theory to compute jumps in the charges, and one can make use of the fact that K-theory contributes an integral index to the path-integral to argue that the jumps are anomaly free [7]. This leaves us with one constant still to be determined, the bare charge of the orientifold plane (for example, an $O3^{-}$-plane starts with charge $-1/4$). Although some earlier work on this exists [37], there does not yet exist a satifactory answer within the K-theory framework.

Note added: the classification of consistent and inequivalent O6-plane configurations (for $G_0 = 0$) in subsection 5.3 also appeared in [38].

A. AHSS in complex and real K-theory

In this appendix we will summarize the notion of the Atiyah-Hirzebruch spectral sequence (AHSS), which can be used to compute K-theory groups which have a triangulation. We will show how this generalizes to real variants of K-theory. The first step in using the AHSS is to define an associated graded complex.

A.1 Defining the associated graded complex

Consider a space X, with a triangulation $\hat{X}$. X_p is the subset of this triangulation with simplices of maximum dimension p. For example the two-sphere, $\mathbb{S}^2$, has a triangulation $\hat{X} = X_2$ equal to

$$\{(x_2x_3x_4), (x_1x_3x_4), (x_1x_2x_4), (x_1x_2x_3)\} \tag{A.1}$$

and so

$$\begin{aligned} X_1 &= \{(x_1x_2), (x_1x_3), (x_1x_4), (x_2x_3), (x_2x_4), (x_3x_4)\}\,, \\ X_0 &= \{x_1, x_2, x_3, x_4\}\,. \end{aligned} \tag{A.2}$$

We are now going to use the triangulation to *filter* the group K(X). Define

$$K_0(X) \doteq K(X),\ K_p(X)|_{1\leq p \leq \mathrm{Dim}(X)} \doteq \{x \,\epsilon\, K(X) \mid x \text{ trivial on } X_{p-1}\}\,. \tag{A.3}$$

Going back to our example $X = \mathbb{S}^2$, we have $K_0(\mathbb{S}^2) = \mathbb{Z} \oplus \mathbb{Z}$. When we wrap a D$p$-brane on this S_2, these two charges correspond to Dp-brane charge and D$(p-2)$-brane charge. Thus in this example, $K_1(\mathbb{S}^2) = K_2(\mathbb{S}^2) = \mathbb{Z}$, reflecting the fact that these groups capture only D$(p-2)$-brane charge.

For a finite dimensional space X of dimension N, the K_p's define a *filtration*,

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_N \,, \tag{A.4}$$

which makes $K(X)$ a *filtered complex*. This allows us to define an *associated graded complex*

$$\mathrm{Gr}K \doteq \bigoplus_{p=0}^{p=N} K_p/K_{p+1} \doteq \bigoplus_{p=0}^{p=N} G_p \,. \tag{A.5}$$

Although $\mathrm{Gr}K(X)$ doesn't capture all the structure of $K(X)$, there is, as it turns out, a simple algorithm for computing $\mathrm{Gr}K$ which usefully connects with much of the physicist's intuition about fields as seen in cohomology. Also note that we can easily extend the notion of the associated graded complex $\mathrm{Gr}K$ to any other K-theory, such as $K^{-1}(X), KR(X), K_{\pm}(X)$, etc.

A.2 Computing GrK using a spectral sequence

To compute $\mathrm{Gr}K$, we are going to use a sequence of successive approximations. The first approximation to G_p we can label as E_1^p, and it consists of K-classes on the p-simplices σ_p^i trivial on their boundaries. In other words,

$$E_1^p = K(\oplus_i \sigma_i^p, \oplus_i \dot{\sigma}_i^p) = \oplus_i K(\sigma_i^p, \dot{\sigma}_i^p) \,. \tag{A.6}$$

Of course, since $K(\sigma_i^p, \dot{\sigma}_i^p) = \widetilde{K}(\mathbb{S}^p) = K^{-p}(pt)$ we get that

$$E_1^p = C^p(X, K^{-p}(pt)) \,. \tag{A.7}$$

Since we our using complex K-theory for now, only the even-dimensional co-chains appear.

Now the first approximation gives us a candidate K-classes on the σ_i^p's, but we need to make sure that they can be extended to K-classes on $\hat{X}$ and thus X. The first step is to extend a class on the p-skeleton to a class on the $(p+1)$-skeleton, i.e. take every element in E_1^p and extend it to the σ_i^{p+1}'s. Summing up the integers from each component of the boundary $\dot{\sigma}_i^{p+1}$ yields, with appropriate orientation, a winding number which if non-zero acts as an obstruction to extending the co-chain to the $(p+1)$-skeleton. This is precisely the cohomology differential which maps

$$d_1 : C^p(X, K^{-p}(pt)) \;\longrightarrow\; C^{p+1}(X, K^{-p}(pt)), \tag{A.8}$$

which implies that the next approximation to G_p should be

$$E_2^p = H^p(X, K^{-p}(pt)) \,. \tag{A.9}$$

To continue with this construction, we define elements

$$E_2^{p,q} = H^p(X, K^q(pt)) \text{ with } E_2^p = E_2^{p,-p} \,, \tag{A.10}$$

where p ranges from 0 to the dimension of X, but q is allowed to range freely. With these elements, the whole approximation scheme can then be refined recursively as follows. We define d_r to be the obstruction to lifting a K-class in E_r up r steps. Formally,

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-(r-1)} . \tag{A.11}$$

We then define E_{r+1} as the cohomology of the operator d_r and repeat. We have already seen what E_1, d_1 and E_2 correspond to. In complex K-theory, only odd differentials appear, and d_3 is the Steenrod square Sq^3. For finite dimensional spaces, this procedure terminates at some finite order n. We then have

$$K_p(X)/K_{p+1}(X) = G_p(X) = E_n^{p,-p}(X) . \tag{A.12}$$

There are two features to this construction that are of particular interest. The first is that cohomology appears at the E_2 level in the scheme, hence we can view this construction of K-classes as a sort of refinement of $H^*(X)$ with a different addition structure. The second is that the same coefficients $E_r^{p,q}$ can naturally be used to compute the associated graded complex for other K-theories (as long as there is no involution on the space X) such as $K^{-s}(X)$ and $KO^{-s}(X)$ using a slight modification of the coefficients in eq. (A.10). For example

$$G_p^{-s}(X) \doteq K_p^{-s}(X)/K_{p+1}^{-s}(X) = E_n^{p,-(p+s)}(X) . \tag{A.13}$$

Note that the differentials d_r all map from the $E^{p,q}$'s for $K^{-s}(X)$ to those for $K^{-s+1}(X)$ just as the regular cohomology differential d maps co-chains in $C^s(X)$ to co-chains in $C^{s+1}(X)$.

A.3 Extending the AHSS to K-theories with freely acting involutions

In order to extend the AHSS construction above to K-theories with a freely acting spatial involution, e.g. $KR^{-s}(\mathbb{S}^{p,0})$, we need to make some restrictions on the triangulation, namely that every simplex σ_i^p is mapped by the involution χ to another simplex $\overline{\sigma_i}^p$ such that

$$\sigma_i^p \,\cap\, \overline{\sigma_i}^p = \emptyset . \tag{A.14}$$

As an example, we can give a simple such triangulation for the $\mathbb{S}^{p,0}$. Start with the set of points $\{x_\pm^1, \ldots, x_\pm^p\}$. The simplices that make up the triangulation consist of all sets S of these points (with appropriate orientations) such that

$$x_\pm^i \in S \;\Rightarrow\; x_\mp^i \notin S . \tag{A.15}$$

For example, $(x_+^1\, x_-^2\, x_+^3)$ is a 2-simplex in this triangulation, but $(x_+^1\, x_-^2\, x_-^1)$ is not. We then define the involution to to take

$$\chi :\; (x_\pm^i, \ldots, x_\pm^j) \longrightarrow (x_\mp^i, \ldots, x_\mp^j) . \tag{A.16}$$

which clearly satisfies eq. (A.14).

Given a proper triangulation then, we can continue to follow the procedure above, except now there should be a slight modification of the E_1 term given in eq. (A.6). For example, for computing KR-groups we use

$$E_1^p = KR(\oplus_i(\sigma_i^p \oplus \bar{\sigma}_i^p), \oplus_i(\dot{\sigma}_i^p \oplus \dot{\bar{\sigma}}_i^p)) = \oplus_i KR(\sigma_i^p \oplus \bar{\sigma}_i^p, \dot{\sigma}_i^p \oplus \dot{\bar{\sigma}}_i^p)\,. \tag{A.17}$$

At this point it useful to bear in mind that generator of $K(\sigma^i, \dot{\sigma}^i)$ is real (or pseudo real) for $i = 0 \bmod 4$ but not for $i = 2 \bmod 4$ (the conjugate spinors have opposite chirality) . This implies that for KR- and KH-groups,

$$E_1^{p,q} \;=\; C^p(X|_\chi, \mathbb{Z}) \text{ for } q = 0 \mod 4 \tag{A.18}$$

$$E_1^{p,q} \;=\; C^p(X|_\chi, \widetilde{\mathbb{Z}}) \text{ for } q \;=\; 2 \mod 4 \tag{A.19}$$

$$E_1^{p,q} \;=\; 0 \text{ for } q \text{ odd} \tag{A.20}$$

where $\widetilde{Z}$ refers to the *local coefficients* on X which switch signs with the map χ, i.e. the co-chain takes opposite value for opposite simplices in the covering space X.

The differential d_1 is identical with the usual differential operator, so E_2 gives twisted and un-twisted cohomologies. On the other hand, we can see from eq. (A.18) that d_3 takes twisted chains to un-twisted chains and vice-versa and so is not the Steenrod square like in the complex case. It provides for a difference between KR- and KH-groups.

A.4 Illustrative example: $KR(\mathbb{S}^{6,0})$ and $KH(\mathbb{S}^{6,0})$

To illustrate the AHSS in the context of KR-groups and KH-groups, we can examine as an example the groups $KR(\mathbb{S}^{6,0})$ and $KH(\mathbb{S}^{6,0})$. The E_2 terms are:

$$\begin{aligned}
E_2^{0,0}(\mathbb{S}^{6,0}) &= H^0(\mathbb{RP}^5, \mathbb{Z}) = \mathbb{Z} \\
E_2^{1,-1}(\mathbb{S}^{6,0}) &= 0 \\
E_2^{2,-2}(\mathbb{S}^{6,0}) &= H^2(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) = 0 \\
E_2^{3,-3}(\mathbb{S}^{6,0}) &= 0 \\
E_2^{4,-4}(\mathbb{S}^{6,0}) &= H^4(\mathbb{RP}^5, \mathbb{Z}) = \mathbb{Z}_2 \\
E_2^{5,-5}(\mathbb{S}^{6,0}) &= 0\,.
\end{aligned} \tag{A.21}$$

There are two possible places where the differential d_3 can act non-trivially:

$$d_3 : \; E_2^{0,\,0} \; (\mathbb{S}^{6,0}) = \mathbb{Z} \; \to E_2^{3,-2}(\mathbb{S}^{6,0}) = \mathbb{Z}_2 \tag{A.22}$$

$$d_3 : \; E_2^{1,-2}(\mathbb{S}^{6,0}) = \mathbb{Z}_2 \to E_2^{4,-4}(\mathbb{S}^{6,0}) = \mathbb{Z}_2\,. \tag{A.23}$$

Let us look at the case (A.22) more closely. The generator of $E_2^{0,\,0}(\mathbb{S}^{6,0})$ is a bundle of rank one over the 0-skeleton. Being in E_2 (and thus E_3) means that it extends to a rank one bundle over the 2-skeleton. Let's try to extend it to a rank

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one bundle over the 3-skeleton. Consider the 3-chain in $C_3(\mathbb{RP}^5, \widetilde{\mathbb{Z}})$ which looks like an $\mathbb{RP}^3$ inside $\mathbb{RP}^5$. It has as a boundary a 2-chain made up of two $\mathbb{RP}^2$'s (which explains why $H_3(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) = 0$). In KR-theory, the constant rank one bundle over the $\mathbb{RP}^2$'s can be extended to the "$\mathbb{RP}^3$", but in KH-theory the rank one bundle over $\mathbb{RP}^2$ has non-trivial first Chern class and so cannot be extended to the "$\mathbb{RP}^3$". Looking at eq. (A.22), this fact implies that d_3 takes $z \to 0$ in $KR(\mathbb{S}^{6,0})$ but that $z \to z \bmod 2$ in $KH(\mathbb{S}^{6,0})$. Further analysis of the same kind reveals that d_3 is only non-trivial in eq. (A.23) for $KH(\mathbb{S}^{6,0})$. The final result is then:

$$\begin{aligned} KR(\mathbb{S}^{6,0}) &= \mathbb{Z} \oplus \mathbb{Z}_2 \\ KH(\mathbb{S}^{6,0}) &= 2\mathbb{Z}\,. \end{aligned} \tag{A.24}$$

where the factor of 2 denotes the fact that only even rank constant bundles are allowed.

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